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ORIGINAL PAPER

Application of a stochastic differential equation to the prediction of shoreline evolution

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Abstract Shoreline evolution due to longshore sediment transport is one of the most important problems in coastal engineering and management. This paper describes a method to predict the probability distributions of long-term shoreline positions in which the evolution process is based on the standard one-line model recast into a stochastic differential equation. The time-dependent and spatially varying probability density function of the shoreline position leads to a Fokker–Planck equation model. The behaviour of the model is evaluated by applying it to two simple shoreline configurations: a single long jetty perpendicular to a straight shoreline and a rectangular beach nourishment case. The sensitivity of the model predictions to variations in the wave climate parameters is shown. The results indicate that the proposed model is robust and computationally efficient compared with the conventional Monte Carlo simulations.

Keywords Probability density distribution · Shoreline erosion · Longshore transport · Wave distribution · Fokker–Planck equation

List of Symbols

a_1	Constant in the expression for q_{10} (m^2 s^2/kg)]
C_{gb}	Wave group celerity (m/s)
C_v	Coefficient of variance, $0 \leq C_v \leq 1$, defined as the ratio of the standard deviation to the mean
d	Difference operator
$\boldsymbol{D}$	Variance vector
d_b	Breaker depth (m)
d_c	Closure depth (m)
D_{ij}	*i*th row and *j*th column component of $\boldsymbol{D}$
E	Expectation operator
E_b	Wave energy evaluated at the breaking point (kN)
g	Acceleration due to gravity (m/s^2)
$\boldsymbol{G}$	Deterministic operator of Itô equation
h_b	Discrete wave height at breaking (m)
H_b	Wave height at breaking (m)
H_{bm}	Mean of H_b
H_s	Significant wave height (m)
i	Alongshore cell number
j	Cross-shore node number
k	Time step of shoreline model
m	Time step of FP model
k_1	Sediment transport coefficient or dimensionless empirical coefficient
K_m	Mean of H_w
l	Distance alongshore (m)
m_y	Number of the discrete distribution in cross-shore direction
N	Normal distribution
n_d	Dimension of state space
n_e	Number of random parameters
n_s	Void ratio of sand
p	Probability density function
p_0	Initial of p
q_1	Volumetric longshore sediment transport rate (m^3/s)
q_{10}	Amplitude of volumetric longshore sediment transport rate (m^3/s)

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t	Time (s)
T	Transpose of a matrix
t_0	Initial time (s)
t_k	Time in kth step of one-line shoreline model (s)
t_m	Time in mth step of Thomas algorithm (s)
$\boldsymbol{W}$	Wiener process vector
W_{b}	Gaussian White noise of wave initial breaking angle
W_i	ith component of Wiener process $\boldsymbol{W}$
W'	Standard Gaussian White noise process of H_{b}
x	Distance alongshore (m)
y	Shoreline position in cross-shore direction (m)
$\boldsymbol{y}$	Discrete stochastic process vector
Y	Stochastic process or random variable of shoreline position or random state (which is a component of the state vector)
$\boldsymbol{Y}$	Stochastic process vector
y_0	Initial y
Y_0	Initial random state or random variable of shoreline position Y
$\boldsymbol{Y}_0$	Initial $\boldsymbol{Y}$
Y_{F}	Extension distance seaward from the rectangular beach
y_i	ith component of $\boldsymbol{y}$
y_{m}	Mean of y
Y_{m}	Expected value or mean value of Y
$y_{\max}$	Maximum discrete shoreline position
$y_{\min}$	Minimum discrete shoreline position
y_{s}	Standard deviation of y
$\mathbf{y}_{\mathrm{s}}$	Standard deviation of $\mathbf{y}$
$\dot{\boldsymbol{Y}}$	Time derivative of $\boldsymbol{Y}$
α_0	Incident angle of breaking waves relative to x (°)
$\alpha_{0\mathrm{m}}$	Mean of incident angle of breaking waves relative to x (°)
α_{b}	Wave angle at the point of breaking (°)
γ	Ratio of wave height to water depth at breaking, $= H_{\mathrm{b}}/d_{\mathrm{b}}$
Δt_k	Time increment of one-line shoreline model (s)
Δt_m	Time increment of Thomas algorithm (s)
Δx	Size increment in alongshore direction [m]
Δy	Size increment in cross-shore direction [m]
$\boldsymbol{\eta}$	Operator vector of determination of a dynamical state, may be determined by an appropriate deterministic shoreline evolution model
η_i	ith component of the operator $\boldsymbol{\eta}$
$\boldsymbol{\kappa}$	Vector of drift
κ_i	ith component of $\boldsymbol{\kappa}$
λ	Ratio, $= \frac{\Delta t_m}{\Delta t_k}$
ρ_{s}	Mass density of the sediment grains (kg/m^3)
ρ_{w}	Mass density of water (kg/m^3)
ϕ	$= \frac{1}{8}\rho_{\mathrm{w}} g^{3/2}\gamma^{-1/2} a_1 \frac{1}{d_{\mathrm{c}}} \frac{\partial \sin(2\alpha_{\mathrm{b}})}{\partial x}$
ψ	$= \frac{\partial\left(2\alpha_{0\mathrm{m}} - 2\arctan\left(\frac{\partial y}{\partial x}\right)\right)}{\partial x}$
ω	$= \frac{1}{8}\rho_{\mathrm{w}} g^{3/2}\gamma^{-1/2} a_1 \frac{1}{d_{\mathrm{c}}}$

1 Introduction

Longshore sediment transport and shoreline evolution have long been serious concerns for coastal scientists. There is an urgent need to develop models that can predict how the coastline morphologies might change a few years hence because this knowledge is essential for coastline management, environmental impact assessment, and the design of coastal structures. The complexity of the coastline system and the extremely wide spatial and temporal ranges of nonlinearly interacting wave, tide, and current phenomena are reflected in the complexity of the shoreline change models. The ability to predict changes in coastal engineering is hampered by rarely and poorly measured quantities and by the prohibitive computational demands of applying deterministic dynamical equations for fluid flow and sediment transport over relatively short periods of a single storm (de Vriend et al. 1993). Much research has therefore focused on simplified models for longshore transport. The simplest model is the one-contour line shoreline evolution model, which is popular for investigating the long-term evolution of the plan shape of beaches. This model has been extensively studied for some time and has proved useful in modelling the dynamics of a wide range of coastal morphologies because it often captures the essence of complicated interactions (Pelnard-Considère 1956; Hanson and Kraus 1989; Dean and Grant 1989). However, quite often, discrepancies exist between the field data and the theoretical predictions (Bayram et al. 2001; Davies et al. 2002; Pilkey et al. 2002), resulting in considerable uncertainties. In other words, the model parameters may not be exactly defined. When the variation is smaller relative to the 'real' values, the dynamic response could be adequately obtained deterministically. However, for coastlines that require precise positions, the variations or uncertainties in the 'real' values of coastal profiles and the environmental conditions may be too large to ignore.

The complex hydrodynamic forcing that results from extremely wide spatial and temporal ranges of nonlinearly interacting waves, tides and currents can cause the shoreline to move randomly and exhibit considerable variability at various temporal and spatial scales, as discussed by Larson and Kraus (1994), Stive et al. (2002) and Reeve and Spivack (2004). Deterministic shoreline evolution models are incapable of manifesting any variability in change rates that might occur in this morphology (Cowell et al. 2006; Camfield and Morang 1996), which strongly suggests that the deterministic change model is not sufficient to model the evolutionary dynamics of many morphologies. In many cases, incorporating some type of variability or uncertainty into the change process of the coastline is needed. For example, Walton (2007) discussed the importance of the spatial variability of both the shoreline diffusivity coefficient and the wave angle in predicting shoreline evolution. A more rational theoretical framework is to treat the dynamical response of the shoreline over time as a time-dependent stochastic system with probabilistic predictions of the shoreline changes at any future time as the modelling goal.

In the past decade, various probabilistic models have been proposed for predicting shoreline evolution. Some of these models dealt with the variability of hydrodynamic input parameters, such as wave height and wave period (Vrijling and Meijer 1992; Southgate 1995; Ruggiero et al. 2006), while others focused on the variability in model parameters (Cowell et al. 2006). As these models are intended to offer an objective and quantitative method for analysing the behaviour of real shoreline systems, they should contain an adequate description of the physical processes and be computationally efficient for long-term simulation. Therefore, nearly all existing probabilistic shoreline evolution models are based wholly or in part on the one-line model originally proposed by Pelnard-Considère (1956). This type of one-line governing equation may be solved many times numerically with specific initial conditions at time t_0 by generating wave conditions from a random-number generator (i.e., Monte Carlo sampling). The results are a bundle of trajectories in phase space, all originating from the same point at t_0. Vrijling and Meijer (1992), in a notable earlier probabilistic modelling study, performed both simple risk analysis by varying the longshore transport rate and full Monte Carlo simulations of the shoreline positions. Dong and Chen (1999, 2000) extended their modelling approach to include random temporal variability and storm beach profile changes due to cross-shore sediment transport to account for the influence of cross-shore transport on the transient shoreline position to determine the true risks of shoreline erosion over the long-term, but an explicit expression of probability density function (PDF) of shoreline positions is neglected. In studying the recession of a soft cliff, Hall et al. (2002) presented an episodic stochastic cliff erosion model, which uses random sampling of the input parameters of the size and time of cliff recession events from probability distributions (from a Monte Carlo simulation) to represent the uncertainty in the recession process, in which the output is expressed as a probability distribution. Ruggiero et al. (2006) used a one-line coastal evolution model in conjunction with a wave transformation model to investigate the probabilities of decadal shoreline changes along the Washington coast using 1,121 combinations of wave height, period and direction.

Besides, Southgate (1995) addressed the effects of a full range of possible sequences of wave conditions (wave chronology effects) on the seabed profiles or beach levels by running the COSMOS-2D morphodynamic profile model developed at HR Wallingford. Cowell et al. (2006) used the closure depth as a random input to predict the potential changes in shoreline, as generated from 1000 simulations using a Shoreface Translation Model (Cowell et al. 1992, 1995). However, Monte Carlo simulations can raise significant computational issues, particularly if the number of random variables is large or an accurate determination of the tail distribution is required. Some efforts have been made to simplify the problem and to produce approximate solutions without the need for computationally intensive Monte Carlo simulations. Benassai et al. (2001) presented a level II probabilistic model for predicting the lifetime of a beach nourishment project using the beach planform model and applied the model to assess the effects of the project's geometry and the fill material on the probability of failure through a sensitivity analysis. Reeve and Spivack (2004) developed a statistical-dynamical morphological method that directly solved for the statistical moments of the shoreline position and presented the time-dependent ensemble averaged solutions for a given wave climate. However, this method does not give direct information on the probability distribution of the shoreline position.

When random quantities of the wave conditions are imposed as coefficients, which are defined in terms of the probability distributions, the quantities of concern for describing the response of the coast, including the locations and changing rates, will fluctuate in a stochastic way. The mean evolutionary dynamics are driven, using the deterministic model as a platform when a probability distribution is available, to obtain the 'real' position at any time t with a known initial position y_0 at an initial time. Fortunately, this concept, which includes the probabilities for describing the shoreline's evolutionary position and the behaviour of the evolution rate, can be stimulated in the same way solving a stochastic differential equation (SDE) in physics (Soong 1973; Gardiner 2004).

The theory of SDEs can be traced back to the works by Einstein (1905) and von Smoluchowski (1906) in their attempts to explain Brownian motion phenomena. Based on Gardiner's comment (2004), for all practical purposes, Einstein's effort marked the beginning of stochastic modelling of natural phenomena. A few years after Einstein's achievement, Langevin attacked the problem much more directly and produced the first example of an SDE that attempted to model the dynamics of such motion in terms of differential equations. In the modern physics literature, SDEs commonly consist of Langevin equations. Itô (1951) formulated a rigorous theory of SDEs based on the particular concepts of stochastic integrals and differentials and suggested that the input of the general SDE could be represented by white Gaussian noise or Poisson noise, such that diffusion, drift and jumps can occur (Gardiner 2004). Itô's calculus effectively provided a rigorous basis for Langevin's approach to Brownian motion that, until that time, had been lacking. In connection to applications of SDEs in hydrology engineering, Bodo et al. (1987) summarized fundamentals such as Markov processes, Itô's calculus, and forward and backward Kolmogorov equations. Van der Berg et al. (2005) introduced uncertainties to the biochemical oxygen demand model by adding white noise processes. The modelling of random wave forces using a white stationary stochastic process to define the Itô differential equation was investigated by Sobczyk (1991), Dostal and Kreuzer (2011) and Vanem et al. (2012).

In this work, a SDE model of shoreline evolution is developed in which the physical process is formulated based on the one-line type model and the random effects from wave height are characterised by a Gaussian white noise process as a first approximation to the true distributions of these effects. The SDE can then be solved by the Monte Carlo simulation method (Kloeden and Platen 1992). Alternatively, its solutions can be described by the Fokker–Planck (FP) equation (Risken 1984) as Markovian diffusion processes (Horsthemke and Lefever 1984). The latter provides an easier explanation of the physical meaning of each term in the equation and will be discussed here. The FP equation is a partial differential equation for the probability density and the transition probability of these stochastic processes. The PDF $p(y, t)$ for the shoreline position can then be obtained by solving the FP equation, thus shifting a random shoreline evolution problem to the solution of a deterministic partial differential equation, which greatly simplifies the analysis.

The paper is organised as follows. In Sect. 2, the basis of the deterministic one-line model is provided briefly, and the shoreline evolution is formulated in a general SDE form as a Markov stochastic diffusion process, which leads to a FP model to determine the PDF of shoreline position. Section 3 presents the numerical implementation and solving procedure and its application to two idealised cases: a long impermeable Jetty case and a rectangular beach fill case. The accuracy and robustness of the model are evaluated by comparing them against results from Monte Carlo simulations, while some features of the PDF of the shoreline position are given through a sensitivity analysis. The derivation of the multi-dimensional SDE shoreline evolution model is also discussed. Conclusions are given in Sect. 4.

2 Formulation of stochastic shoreline model

2.1 One-line model

The coastal literature contains many deterministic shoreline evolution models, among which the one-line model is the simplest and most widely used in engineering analysis and design. The theoretical basis for the model can be stated as follows. Under the actions of waves and currents, beach profiles undergo continuous changes. Such changes can occur gradually, due to the cumulative effect of the spatial gradients in the longshore transport rates, or rapidly, as the result of large storms. Over a given time period, the time-dependent beach recession is the result of both processes (Dong and Chen 1999, 2000). However, for many sandy beaches, although changes in the beach profile during a storm are large, the beach has the tendency to recover during normal wave conditions so that their influence on the long time-averaged position of the shoreline is small compared with the cumulative effects of the longshore transport gradients. As will be shown in the next section, this model provides a convenient framework to allow an analogous stochastic treatment to be developed in a straightforward manner.

Formulations of such phenomena typically follow one of several expressions within the one-line theory (Pelnard-Considère 1956; US Army Corps of Engineers 1984; Hanson and Kraus 1989; Dean and Grant 1989; Cowell et al. 1992; Kamphuis 2000), which is based on the assumption that the cross-shore profile shape remains unchanged as the shoreline position varies. The evolution processes for a shoreline position with the rate η can then be described by:

$$\frac{\partial y(x,t;\eta)}{\partial t} = \eta(y(x,t;\eta),t) \tag{1}$$

where y is the shoreline position and x is the shoreline location along a coastline. This corresponding rate may be attributed to the effect of wave climate or sediment characteristics.

The one-line model originally proposed by Pelnard-Considère (1956) has become more popular in deterministically predicting the long-term shoreline evolution. This model assumes that the beach profile moves in parallel to itself, while maintaining its shape at the specified time scale of description (several months to many years). The temporal changes in the shoreline position are linked to the spatial gradient of the sediment transport rate

$$\eta(y(x,t;\eta),t) = -\frac{1}{d_c}\frac{\partial q_l(x,t)}{\partial x} \tag{2}$$

where q_l is longshore sand transport rate, x is the spatial coordinate along the axis parallel to the trend of the shoreline, y is the shoreline position, and d_c is the depth of closure. The general expression for q_1 may be given by:

$$q_1(x,t) = q_{10}(x,t)\sin[2\alpha_b(x,t)] \tag{3}$$

where q_{10} is the amplitude of longshore transport rate and α_b is the breaking wave angle to the local shoreline x given by:

$$\alpha_b(x,t) = \alpha_0(x,t) - \arctan\left(\frac{\partial y}{\partial x}\right) \tag{4}$$

where α_0 is the incident breaking wave angle relative to the x axis. In general, q_{10}, α_0 and α_b may all vary in time and along the shore. α_b is primarily determined by the local configuration of the coastline.

For predictive purposes, an expression for q_{10} in terms of wave parameters and sediment properties is required. One of the most widely recognised expressions is the CERC formula (US Army Corps of Engineers 1984):

$$q_{10} = (E_b C_{gb})a_1 \tag{5}$$

where a_1 is a constant given by $a_1 = \frac{k_1}{2(\rho_s - \rho_w)(1-n_s)g}$, E_b and C_{gb} are the wave energy and group velocity evaluated at the breaking point, ρ_s is density of sand, ρ_w is density of water, n_s is the void ratio of sand, and k_1 is an empirical constant. Adopting the linear shallow water wave approximation gives $E_b = \frac{1}{8}\rho_w g H_b^2$, where g is the gravity acceleration and $C_{gb} = \sqrt{gd_b}$. Further assuming $H_b \approx \gamma d_b$, where d_b is the breaker depth, Eq. (5) then becomes:

$$q_{10} = H_b^{5/2}\left(\frac{1}{8}\rho_w g^{3/2}\gamma^{-1/2}a_1\right) \tag{6}$$

From the above equations, it is clear that the shoreline change rate is dependent on the two breaking wave parameters, H_b and α_0, the sediment transport coefficient k_1, and the depth of closure d_c. The breaking wave parameters can be calculated from the offshore wave height, period and angle using a wave transformation model, such as SWAN, described by Booij et al. (1999) and Ris et al. (1999).

The reliability of the CERC formula for predicting beach morphology has been discussed over many years (Kamphuis 2000). Greer and Madsen (1978) were early reviewers who recommended the formula to be used for the order estimate only. The focus of the discussions is the value of the constant k_1. Schoonees and Theron (1994) discussed the relation of the value of k_1 with the grain size. Most of the data available for calibrating the empirical one-line formulations is obtained from field measurements. Field measurements in the dynamic surf zone are non-controllable and non-repeatable, which may lead to large uncertainties. Pilkey et al. (2002) stressed that discrepancies often exist between field data and theoretical predictions due to more or less intense environmental fluctuations. Kamphuis (2000) also criticised the overestimates that come from the CERC expression and suggested the sediment transport rates are proportional to H_b^2 and $\{\sin[2\alpha_b(x,t)]\}^{0.6}$. In such situations, researchers should resort to SDEs or other probabilistic models to explore the uncertainty of 'real' solutions. These approaches can reveal how both variability and uncertainty in input conditions is transferred to a range of predicted shoreline positions, which makes it possible to establish confidence limits and determine the statistical distribution of the future shoreline position.

2.2 Formulation of the stochastic shoreline evolution model

Assuming that the evolutionary rate η is one of a collection of admissible change rates $\boldsymbol{\eta}$, the evolution uncertainty is thus introduced into the shoreline by the variability of the change rates. This corresponding phenomenon may be attributed to the effect of wave climate differences or sediment characteristics, such as the randomness in input parameters (breaking wave height, period, and angle) and uncertainties in the model parameters (sediment transport coefficient k_1 and the closure depth d_c, etc.). The evolution rates and position variability may also be affected by the boundary and initial conditions (e.g., initial shoreline position). With this assumption of a family of admissible change rates and the associated probability distribution of the shoreline position at any time, one thus obtains y_m, an expectation of the shoreline, with location x at time t. Hence, the corresponding differential equation for mean change dynamics

$$\frac{\partial y_m(x,t;\eta)}{\partial t} = \eta(y_m(x,t;\eta),t) \tag{7}$$

is one of reasonable descriptions of shoreline changes. The following will discuss how the expectation of the shoreline position via a derivative PDF of the shoreline position can be achieved.

For a long-term perspective, the morphological response to storms is analogous to 'noise' around the long-term trend, which is caused by the fluctuating morphological forcing of the waves, tides and surges. The time-varying uncertainties in shoreline position estimation can be typically a combination of source accuracy (e.g., georeferencing), interpretation error (e.g., field mapping techniques), and natural short-term variability consisting of both shore-term beach changes and variations in water level prior to data collection (Ruggiero and List 2009). Additionally, the geographical surface features are generally determined from remotely sensed data and field measurements. The outcomes of these assessments can raise matching and omission errors, as discussed by Fernandes da Silva and Cripps (2008). Thus, the initial shoreline position y_0 is treated as a random variable to account for the uncertainties caused by the short-term variations in either the bed levels or water levels in extracting the initial shoreline position from chart or survey data.

Assuming the randomness of the longshore transport is entirely due to the randomness in the breaking wave height means that the empirical coefficient k_1 and the closure depth d_c and α_0 of above one-line model are taken as constants in formulating the simple SDE model; these parameters can also be treated as random, which will be discussed later. As the longshore transport rate is explicitly dependent on $H_b^{5/2}$, it is convenient to introduce another random parameter defined as $H_w = H_b^{5/2}$.

Therefore, the shoreline position y_0 is a random variable at t_0 that considers the uncertainties in measurement and short-term variations. These uncertainties will change over time when the shoreline position evolves over time following with an empirical expression (such as the one-line model). In other words, $y(x,t)$ is one of a collection of stochastic variables dependent on time t; therefore, $y(x,t)$ is a stochastic process. Moreover, the evolutionary shoreline position can, most rationally, be written in the form of differential equations with random initial conditions (Soong 1973; Gardiner 2004):

$$\begin{cases} \frac{dy(x,t)}{dt} = \eta(y, x, H_w) \\ y(x,0) = y_0(x) \end{cases} \tag{8}$$

where η is an operator of the shoreline position evolving over time that is subject to random breaking wave action for a prescribed random initial shoreline position y_0. This dynamic system with the operator and random initial conditions describes the evolutionary shoreline position as a stochastic process rather than a variable one (Sobczyk 1991; Vanem et al. 2012).

Stochastic equations associated with these random processes can be used for a wide class of random excitations with finite correlation time, especially for excitations which can be represented as a response of coastal dynamical systems to a white noise excitation where the Markov process theory can be used (Sobczyk 1991). Substituting the longshore transport model to Eq. (8) gives

$$\frac{dy(x,t)}{dt} = -H_w \phi(y,t) \tag{9}$$

where $\phi(y,t)$ is defined by $\phi(y,t) = \frac{1}{8}\rho_w g^{3/2}\gamma^{-1/2} a_1 \frac{1}{d_c} \frac{\partial \sin(2\alpha_b)}{\partial x}$.

Before admitting key resources of uncertainty from H_w into the above equation, it is worth recalling the method of the description of the statistical characteristics of the wave height. The distributions of the characteristic wave parameters are usually obtained either from measurements or from hindcast waves based on wind data. The distributions of significant wave height have been shown to be well modelled by a log-normal distribution (Jasper 1956; Guedes Soares et al. 1988) and a Weibull distribution (Battjes 1972; Guedes Soares and Henriques 1996). Guedes Soares and Ferreira (1995) proposed a parametric model for the long-term data that adopts the Box-Cox transformation (Box and Cox 1964) to transform the data set into a normal one and then fits the transformed data to a normal distribution to reduce the introduction of additional uncertainties from fitting the data with various parametric distributions. Other distributions have also been used to fit significant wave height (Guedes Soares 2003). Although real data seldom perfectly support the Gaussian assumption for the long-term trends of significant wave height, the theory associated with the Gaussian process is very well understood (Gardiner 2004) and has a vast amount of available tools (Asmussen and Glynn 2007; Rychlik et al. 1997), which led to the decision to adopt a Gaussian white noise model as a first approximation to the random variations of wave excitation. As stated by Sobczyk (1991) and Vanem et al. (2012), the noise forcing term confirms a zero-mean Gaussian distribution, with an identical variance, and assumed independent in space and time. This model may be relatively accurate when only small variability exists in the stationary excitation of breaking wave heights where the physics sets in a limited period (Vanem 2010). We will return to this issue later in the paper.

If the stochastic processes of wave excitation are assumed to act as Gaussian white noise (Sobczyk 1991; Vanem et al. 2012), the random variations of H_w can be split into two parts:

$$H_w = K_m + W(t) \tag{10}$$

where K_m is the mean value of H_w and $W(t)$ is a Gaussian white noise process with a mean of zero and a variance of D, which is defined as $D = C_v K_m$ with $0 \le C_v \le 1$. In other words, the wave height H_w is independently distributed by Gaussian random variables for each t with mean K_m and variance D. The driving process $W(t)$ appears to be the pathwise derivative of a Gaussian white noise process, $dW(t)$, with $E\{dW(t)\} = 0$ and $E\{[dW(t)]^2\} = 2D dt$.

This pragmatic choice of wave excitation with the Gaussian processes is also considered reasonable because the long-term shoreline position distribution is not controlled by the tail distribution of the wave parameter, as in the case of storm-induced erosion, but rather is controlled by the cumulative longshore transport due to the overall wave climate.

After adopting Gaussian white noise in the wave height, the evolutionary shoreline position will be characterised by a Markovian process (Horsthemke and Lefever 1984), which can be written as

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = -[K_\mathrm{m} + W(t)]\phi(y,t); \quad W(t) \sim N(0,D) \tag{11}$$

where $N(0,D)$ denotes for a normal distribution with zero mean and variance D.

The decomposition of Eq. (11) can be recast in the standard form of the stochastic Itô equation (see Soong 1973):

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = -\phi(y,t)K_\mathrm{m} + G(y,t)W(t) \tag{12}$$

where $G(y,t) = \phi(y,t)$. Though $G(y,t)$ and $\varphi(y,t)$ behave as the same expression, maintaining the initial symbol here can be a smart move when describing the individual contributions of the drift term and the diffusion term.

A simple physical interpretation of the model is that the sediment flux consists of two components, a mean flux and a random flux. The former is similar to the original, deterministic one-line model and determines the average evolution rate (the first moment of the rate of change in position), while the latter controls the variability in the evolution rate of the individual shoreline position (the second moment of the rate of change in position). Furthermore, the diffusion Markov process described by the Itô stochastic equations enables us to access the existing analytical and numerical methods for many practical problems (Spencer Jr and Bergman 1993). For instance, the stochastic Itô equation can be performed by Monte Carlo simulations (Kloeden and Platen 1992) to obtain the shoreline trajectories, which is out of great interest here. As discussed by Soong (1973) and Gardiner (2004), the stochastic Itô equation is satisfied the following FP equation

$$\frac{\partial p(y,t)}{\partial t} = \frac{\partial}{\partial y}[\phi(y,t)K_\mathrm{m}p(y,t)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[(G(y,t)DG(y,t)^T)p(y,t)\right] \tag{13}$$

to obtain the expressions for the evolutionary PDF of the shoreline position that was carefully derived in (Soong 1973), among numerous other places and was subsequently studied in many references (Risken 1984; Gardiner 2004). The FP equation describes the evolution of the probability distribution moving forward in time and therefore satisfies a conservation law, indicating that no probability is lost in the system governed by SDEs and thus, the probability of the trivial event is still 1 (Sura 2003).

To solve Eq. (13), the following initial condition and boundary conditions must be imposed

$$p(y,t_0) = p_0(y) \tag{14a}$$

$$\phi(y_\mathrm{min},t)K_\mathrm{m}p(y_\mathrm{min},t) - \frac{1}{2}\frac{\partial(GDG^T p(y,t))}{\partial y}\Big|_{y=y_\mathrm{min}} = 0$$
$$\phi(y_\mathrm{max},t)K_\mathrm{m}p(y_\mathrm{max},t) - \frac{1}{2}\frac{\partial(GDG^T p(y,t))}{\partial y}\Big|_{y=y_\mathrm{max}} = 0 \tag{14b}$$

where y_max and y_min are the maximum and minimum shoreline positions, respectively, that the individual section may attain in any given time period. Observe that the boundary condition in Eq. (14), $p(y,t)$, must satisfy the normalisation condition in any given time

$$\int_{y_\mathrm{min}}^{y_\mathrm{max}} p(y,t)dy = 1 \tag{15}$$

With such an approach, the statistical characteristics, such as the mean and standard deviation, of the 'real' coastal positions can be obtained by

$$y_\mathrm{m}(x,t) = \int_{y_\mathrm{min}}^{y_\mathrm{max}} yp(y,t)\mathrm{d}y \tag{16}$$

$$y_\mathrm{s}(x,t) = \sqrt{\int_{y_\mathrm{min}}^{y_\mathrm{max}} (y-y_\mathrm{m})^2 p(y,t)\mathrm{d}y} \tag{17}$$

instead of a single deterministic value. This evaluation yields the mean evolutionary dynamics of shoreline position when running this model for every time increment. Above all, the uncertainties in the change rates are introduced into an entire stretch of coastline in space by applying these evolutionary formulae to each individual section.

The theoretical upper and lower limits, y_max and y_min, need to cover the reasonable distribution range of non-negative probability densities of the shoreline position. The stronger boundary conditions $p(\pm\infty,t) = 0$ are imposed, which implies that the values of y remain finite with a probability of one. This result can be interpreted as having an absorbing barrier placed at infinity for the probability flow. Obviously, this addition will increase the non-necessary computing time due to the increased number of discrete grid points because most of the values are zero. Generalising the previous remark, the rescaled absorbing barriers symmetrically placed at $y = y_\mathrm{m} \pm m_y\Delta y$ for each cell can be imposed, i.e., $p(y_\mathrm{max},t) = 0$ and $p(y_\mathrm{min},t) = 0$; here, $y_\mathrm{max} = y_\mathrm{m} + m_y\Delta y$ and $y_\mathrm{min} = y_\mathrm{m} - m_y\Delta y$. y_m is the mean position and may change over time for each cell along the shore. m_y is a positive integer that can be between 100 and 500, depending on the resolution, to describe the discrete value of the PDF in the cross-shore direction. Thus, y_max and y_min vary over cells and with time.

3 Numerical results and discussions

3.1 Solution procedures

The FP equation Eq. (13) is a deterministic nonlinear partial differential equation for which a closed form solution would be intractable; thus, this equation must be solved numerically (Wehner and Wolfer 1983; Spencer and Bergman 1993; Anderson 1995). In this model, a second-order implicit finite difference scheme is used with derivative terms approximated by backward difference in time and central difference in space. The general calculation steps for each time increment are as follows:

Step 1 Determine the initial condition, $p(y_j,0)$, $y_j = j\Delta y$ $(j = 1, 2, 3, \ldots 2m_y + 1)$; here, m_y is an integer that defines the absorbing barriers, $2m_y + 1$ is total number of nodes in the cross-shore direction and Δy is the grid increment. This integer can be tentatively chosen based on a large upper limit for y calculated by a deterministic computation before running this model. $p(y_j,0)$ represents the initial distribution of the shoreline, and the mean of the initial shoreline can then be calculated as $y_{\mathrm{m}}(0)$.

Step 2 Calculate the shoreline position $y(x_i,t_k)$ based on the one-line model, Eq. (2). Here, x_i denotes the position of the ith cell $(i = 1,2,\ldots)$ and $t_k = k\Delta t_k$ $(k = 0,1,2,\ldots)$, where Δt_k is the time increment. The mean shoreline position $y_{\mathrm{m}}(x_i,t_k)$ is then obtained according to Eq. (16), which is used to update the local breaking angle and the longshore transport rate.

Step 3 Calculate $\varphi(y_j, t_k)$ and $G(y_j,t_k)$.

Step 4 Solve the FP equation (13) for $p(y_j,t_m)$ by the Thomas algorithm (Wang and Anderson 1982), where $t_m = m\Delta t_m$ $(m = 0,1,2,\ldots)$ and Δt_m is the time step for the FP equation. The time step Δt_m can be chosen as either $\Delta t_m = \Delta t_k$ or $\Delta t_m = \lambda\Delta t_k$, $\lambda > 1$. Both time steps must satisfy the respective Courant stability criteria.

Step 5 Update $y_{\mathrm{m}}(x_i,t)$ and $y_{\mathrm{s}}(x_i,t)$ using $p(y_j,t_m)$ according to Eqs. (16) and (17).

Repeat steps 2 and 5 for the next time step.

3.2 Numerical simulations

The first test case is a straight sandy shoreline $(y_{\mathrm{m}}(x,0) = 0)$ with a long impermeable jetty perpendicular to it at the immediate right of cell 100 with a littoral drift arriving from the left; no by-passing of the jetty occurs. A diagrammatic definition of the main variables is given in Fig. 1. The values used for the constant input variables are

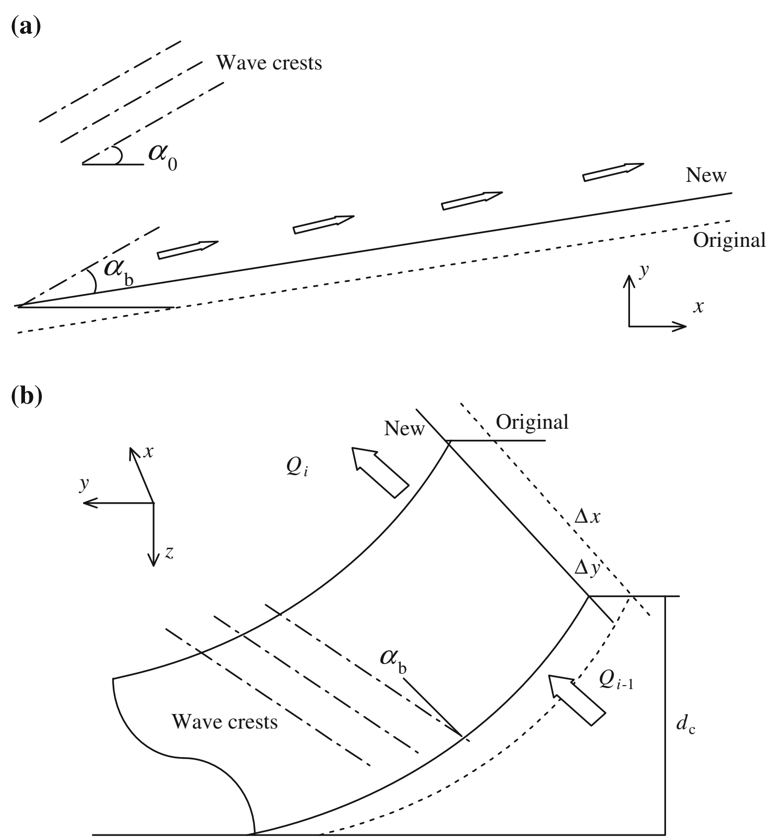

Fig. 1 Definition sketches: **a** wave breaking angle and alongshore shoreline change; **b** depth of closure and sediment transport flux for an individual cell

Table 1 Input variables

Variable	Symbol	Values	Unit
Depth of closure	d_c	11.8	m
Breaking wave angle	α_0	10°	degree
Density of water	ρ_w	1,000	kg/m^3
Density of sand	ρ_s	2,650	kg/m^3
Empirical constant	k_1	0.41	
Gravity acceleration	g	9.8	m/s^2
Breaker index	γ	0.78	
Void ratio of sand	n_s	0.4	

given in Table 1. The simulations are performed with a cell width $\Delta x = 25$ m and a time increment $\Delta t_k = 0.1$ day. A one year simulation of the shoreline evolution is performed for the boundary condition at $x = 2{,}500$ m, setting $\frac{\partial y(0,t)}{\partial x} = \tan \alpha_b$, which indicates that the shoreline has an orientation that allows for no sediment transport because no material can pass the structure, i.e., the shoreline normal is in the direction of the waves. The final boundary condition requires that the derivative goes to zero as the beach becomes straight.

In all simulations, the following variables are used unless stated otherwise (as in the sensitivity tests): K_m is 0.375 and C_v is 0.15; the mean of the initial shoreline y_0 is zero with a standard deviation of 2.5 m in a Gaussian distribution. Considering the maximum deposition distance and to ensure the accuracy of the predictions of the PDFs, $\Delta y = 0.5$ m and $m_y = 150$ are used in the cross-shore direction.

The computed mean shoreline positions y_m after one year are shown in Fig. 2. As expected, on the weather side of the jetty, the shoreline moves seaward as the result of sand accumulation at cell 100 ($x = 2{,}500$ m), which is adjacent to the jetty building the furthest out. Figure 2 also shows the

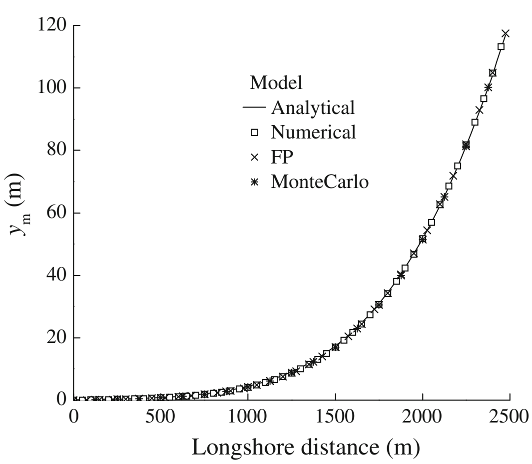

Fig. 2 The shoreline evolution predictions from various models are compared for a theoretical Jetty

deterministic solutions (numerical and analytical) obtained based on the mean value of K_m and using the algorithms in Komar (1983) and Kamphuis (2000) as a comparison. The Monte Carlo simulation is performed and the results are plotted in Fig. 2 (with legend of Monte Carlo) to check the accuracy of the FP code. The differences between these solutions are very small. The solution to this problem typically took approximately 37 min of CPU time on a 3 GHz Intel Pentium D processor using the FP model. To put this in perspective, an efficient Monte Carlo simulation of the same system with 10,000 realisations took approximately 4 h. The FP model can provide the desired randomness as shown in the following graphs.

Figure 3 shows the variations of the calculated y_s along the shoreline at 36.5 days and 365 days. Apart from showing that the randomness of the shoreline position increases with time, the model's results also reveal that, spatially, this increase is nonlinear, with the largest value adjacent to the

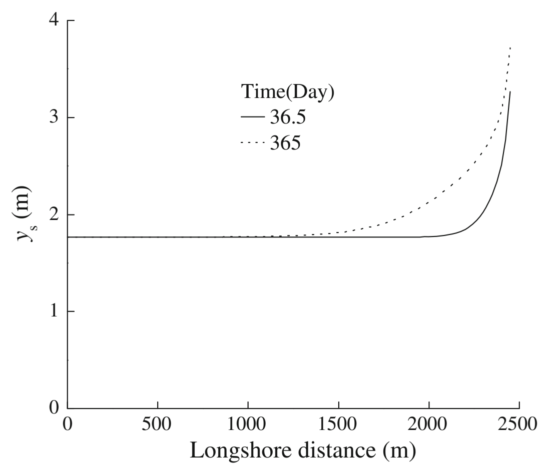

Fig. 3 The distribution of y_s changes with respect to the longshore distance over time

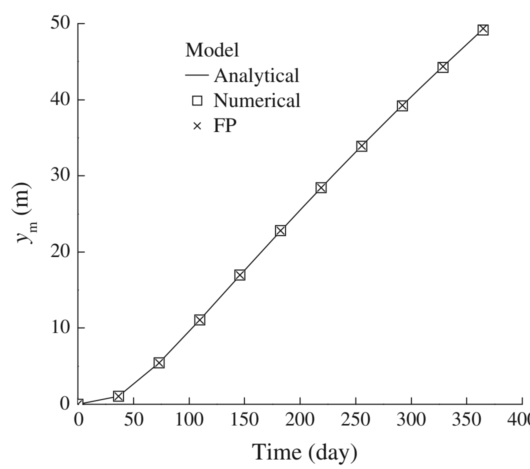

Fig. 4 The various models' predictions of the y_m value show good agreement at cell 80 over time

jetty and the smallest value far from the jetty, reflecting the trend in the gradient of the longshore transport. The reason is that the variance of the shoreline position distribution obtained from this evolution model is determined not only by the variability in the initial shoreline distribution but also by the variability in the affined shoreline evolution term, such that the cells with larger variability experienced more severe excitations. Figure 4 shows the progressive accretion y_m with time at cell 80 ($x = 2{,}000$ m). The results from all models considered follow a very similar trend with the present model, predicting slightly greater values than that by the analytical solution, due to the random sediment flux. The computed means of shoreline position are in agreement with the results investigated by Komar (1983). In this study, the small breaking angle is chosen to satisfy the small angle assumption of the analytical solution.

The variation of $p(y,t)$ over time is shown in Fig. 5 at cell 80. Apart from the expected seaward shift of its peak value due to the accretion process, the probability density curves also become wider over time because of the process of diffusion. It is also clear that, although the process model is nonlinear due to the feedback effect of the changing shoreline orientation, the probability density at any location along the shoreline, i.e., the distribution of the shoreline position, is Gaussian-shaped. This result is as expected because the relationship between the stochastic process of $y(x,t)$ and H_w is linear at any location x.

As discussed earlier, the expectation and variance in the position distribution obtained from this SDE model is associated with the variability in wave height and the variability in the initial position. Simulations are performed with varying values of input parameters to assess the sensitivity of the coastline change. To represent the influence of an increasing mean wave height, $K_m = 0.156$, 0.25, 0.375, and 0.53 are used, and the predicted y_m and y_s at cell 80 and the final step are shown in Fig. 6. As expected, the mean shoreline position y_m and deviation y_s increase

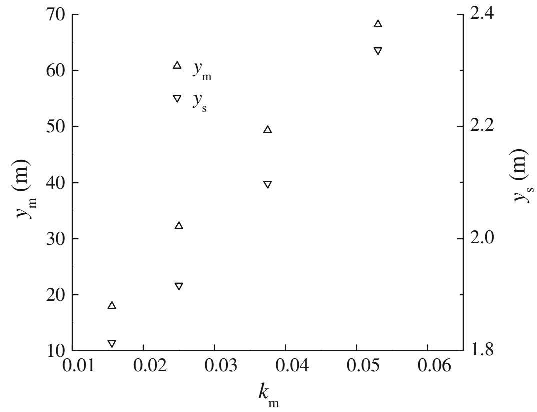

Fig. 6 The predicted y_m and y_s values vary with different K_m input values

sharply with an increase in K_m. Figure 7 shows the same results with $C_v = 0.05$, 0.15, 0.30 and 0.45. y_s now increases with C_v, but y_m decreases nonlinearly. The similar statistical behaviour is also confirmed by the Monte Carlo model, although the data are not shown here due to page limitations.

The stochastic behaviour of the shoreline position could be affected by the choice of the standard deviation (y_{s0}) of y_0 because the model incorporates the initial shoreline position as a Gaussian random variable. Figure 8 shows the mean and standard deviation of the shoreline position in cell 80 at the final time step, with y_{s0} varying from 0 to 6 m. Both y_m and y_s are sensitive to y_{s0} when its value is small. With a further increase in y_{s0} ($y_{s0} \geq 2$ m), the predicted values are only slightly affected by y_{s0}. This behaviour is due to the sensitivity of the model integration to the numerical resolution of the initial PDF of the shoreline y_0. For a given discretisation, Δy, the PDF of y_0 with a larger y_{s0} can be better resolved, resulting in a smoother PDF curve of y_0. When y_{s0} is zero, shown in the left hand side of Fig. 8, the initial

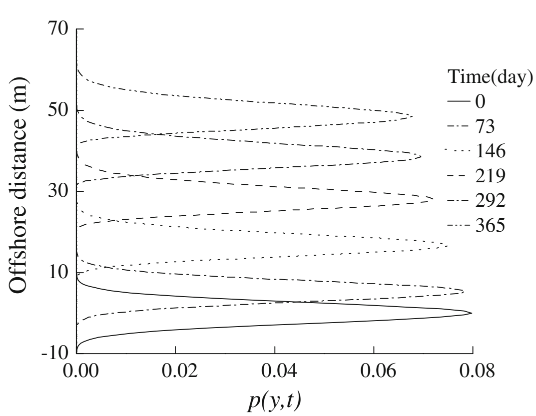

Fig. 5 The typical instantaneous PDFs predict the offshore distance at various times

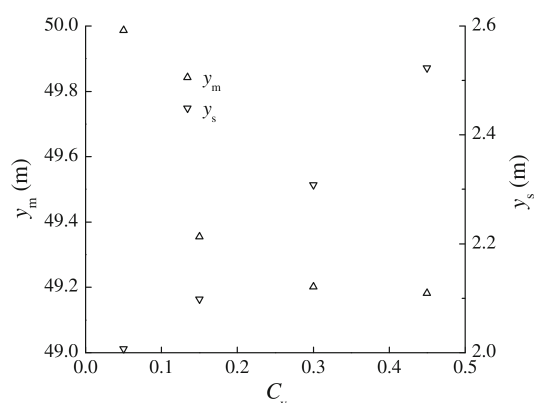

Fig. 7 The values of y_m and y_s vary with C_v

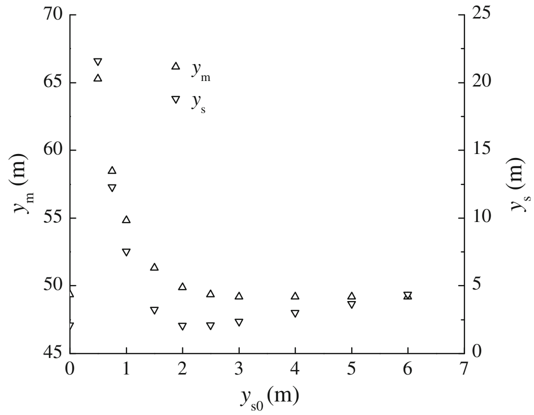

Fig. 8 The values of y_m and y_s vary with the value of y_{s0}

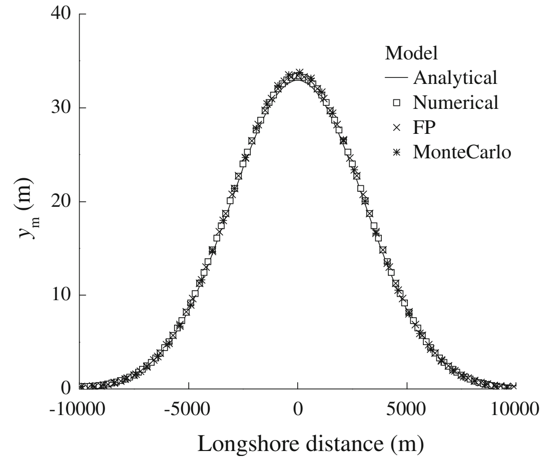

Fig. 9 The predicted shoreline evolution curves for various models are compared for a theoretical beach fill model

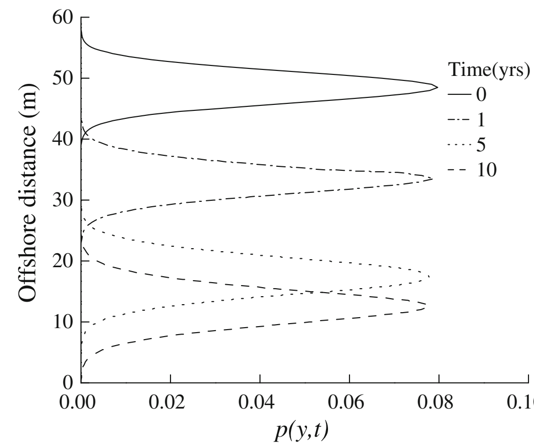

Fig. 10 The typical instantaneous PDFs predict the offshore distance for cell 100 at various long-range times

shoreline position is a fixed constant with no fluctuations (i.e., without considering the influence of initial random input). Thus, the mean value of the shoreline position is quite close to the result by analytical models, and the deviation of the shoreline position is not quite as large as the results from the diffusion term based on the FP equation.

The second example case is that of a rectangular beach fill on an initially straight reach of beach. The difficulties of developing accurate predictions of maintenance nourishment requirement are not surprising due to the complexities of the transport of a placed volume of sand that may not be compatible with the native material and the possibility of a highly variable wave climate. SDEs may provide more qualitative information on the beach profile relative to its maintenance requirements. The fill exists from $-l/2 < x < +l/2$ and extends a Y_F distance seaward from the original beach. Here, $l = 5{,}000$ m, and $Y_F = 50$ m. The simulations are performed with a cell width $\Delta x = 100$ m with 201 cells and are run for a simulated 10 year time period. Other input parameters are the same as for the previous case, with the only difference being the initial wave breaking angle with $\alpha_b = 0$ (i.e., wave approaching normal to shoreline). A similar set-up of this longshore sediment transport model is provided by US Army Corps of Engineers (2002).

Figure 9 shows that the analytical, numerical and FP methods give quite analogous results for the shoreline change at the final step. The developed algorithm successfully solves this problem.

The PDF solution at cell 100 ($x = 0$) for three subsequent periods of time (1, 5, and 10 years) is given in Fig. 10. The PDF is drifting away from its starting point and develops a lower peak and fatter tails as time proceeds. The time-varying behaviour is similar to the Jetty case, and the shape of the PDF is approximately Gaussian.

3.3 Discussion

(1) While the initial results presented here are encouraging and have demonstrated the ability of the linear breaking wave height scheme to predict the uncertainty in the shoreline evolution in response to uncertainties in the wave parameterisation, these results may not be immediately applicable to all shoreline change model studies. One limitation inherent to the technique as developed is the assumption of Gaussian statistics for the distribution of the random wave height values. This assumption is made for pragmatic reasons in that it allows a simple parameterisation of statistics in terms of the mean and covariance matrices and that the transformation of Gaussian variables is conceptually simple. The effects of different types of distributions on the results of the breaking wave height should be investigated before

this model can be applied more widely; it is worth pursuing the outcomes under random input of non-Gaussian distributions, although the associated interpretation of SDEs might be difficult to follow as shown in one of the derivations described by Kontorovich and Lyandres (1995).

(2) Knowledge about the theoretical distribution of wave height parameters is limited; thus, other distribution types are usually more attractive, especially for a long-term statistic of extremes. The distributions of significant wave heights have been investigated by a log-normal distribution (Jasper 1956; Guedes Soares et al. 1988). As suggested by Anderson et al. (2001), a Gaussian distribution can be fitted to the logarithms of the values of H_s. Hence, the logarithms of the H_s values as random variables assuming a Gaussian distribution from this point forward. In other words, if H_s is a log-normal random variable, the distribution function of $\log(H_s)$ is obviously a normal distribution function. This transformation will immediately lead to the antilogarithm or inverse logarithm, such as α_0, into the drift function of the stochastic shoreline evolution equation, requiring the determination of its variability or noise intensity.

(3) The intensity of a white noise process to wave height, $D = C_v K_m$, is a measure of the dispersion of the probability distribution data, which is related to the mean of the wave height. In general, C_v will not be larger than 0.5, even though it can be 1.0. Rosati and Kraus (1991) considered 0.1 or 0.15 accuracy to be associated with uncertainties from instrumentation accuracy and observer bias. Imposing a normal distribution $N(K_m,D)$ for the wave height H_W in the initial joint distribution formulation is not completely reasonable in our motivating application because the intrinsic change rate can then be negative, which results in the position having a positive probability of being negative. However, if D is chosen to be much smaller than K_m, the probability of the intrinsic evolution rate being negative is negligible. For example, even if D is set to be 30 % of the mean K_m, then at least 99 % of samples from $N(K_m,D)$ are expected to be positive.

A standard approach in practice to remedy this problem is to impose a truncated normal distribution $N\left[\vec{K}, \overleftarrow{K}\right](K_m, D)$ instead of a normal distribution; that is, H_W is restricted to some reasonable range $\left[\vec{K}, \overleftarrow{K}\right]$. It should be noted that the stochastic formulation also can lead to the position having a non-negligible probability of being negative when D is sufficiently large relative to K_m. One way to remedy this situation is to set $y(x,t) = 0$ if $y(x,t) < 0$, as stated by Banks et al. (2010), meaning that the shoreline position stays the same size and remains in the system with the possibility to once again change its position. In other words, the flux boundary conditions in Eq. (14a) imposed by the FP model can equivalently hold the solutions properly.

(4) The influences of wave height on the sediment transport rates are assumed to be linear by setting $H_w = K_m + W(t)$ to avoid the definition of a white-noise limit of a system perturbed in the treatment of nonlinear white noise. Ideally, $H_b = H_{bm} + W'(t)$, where H_{bm} is the mean of the breaking wave height and $W'(t)$ is a standard Gaussian white noise process. This treatment reflects better the full nonlinearity of the problem, but it raises a nonlinear multiplicative white noise when the wave height is H_b to the power 2.5, which needs to be achieved by considering the external noise given by the Ornsterin–Uhlenbeck process due to the mathematical impossibility of defining a nonlinear function of white noise. Some treatments, like the approximative evolution operator technique developed by Sancho and San Migue (1980) and the bandwidth perturbation expansion by Horsthemke and Lefever (1980), are available. The latter treatment requires the SDE to be interpreted as a Stratonovic equation rather than an Itô one; additionally, the white-noise limit has to be taken with circumspection by intricate derivation, which will stray slightly from the main point of the introduction of a simple SDE to include randomness in the shoreline change simulations. Furthermore, the multiplicative nonlinear external noise should be studied to elucidate the influence of the white-noise description on the nonlinearity. The reader should consult Horsthemke and Lefever (1984) and Gardiner (2004) for details. In the current theoretical studies, this would make any theoretical comparisons in a reasonable way an extremely formidable task.

(5) The parameter wave angle α_0 appears here in a sine function; thus, it can mathematically handle a random coefficient easily through linearisation of the equation. Actually, the parameter value can be approximated by the first term of its Taylor series if the angle is small, and the SDE Eq. (11) subject to fluctuations from the wave height and the initial breaking angle can be recast by

$$\frac{\mathrm{d}y(x,t)}{\mathrm{d}t} = -K_m\omega(y,t)\psi(\alpha) + \varphi(y,t)\psi(\alpha)W(t) + K_m\varphi(y,t)\psi(0)W_b(t) \tag{18}$$

where $\omega(y,t) = \frac{1}{16}\rho g^{3/2}\gamma^{-1/2}a_1\frac{1}{d_c}$, $\psi(\alpha) = \frac{\partial\left(2\alpha_{0m} - 2\arctan\left(\frac{\partial y}{\partial x}\right)\right)}{\partial x}$ and $\psi(0) = \frac{\partial\left(-2\arctan\left(\frac{\partial y}{\partial x}\right)\right)}{\partial x}$, α_{0m} is the mean of the initial breaking angle. $W_b(t)$ can be approximated by the independent Gaussian white noise fluctuation wave's initial breaking angle. The product of two white noise functions has been neglected in the above equation, similar to the treatment by Unny and Karmeshu (1984). In this case, the corresponding FP equation can be given by:

$$\frac{\partial p(y,t)}{\partial t} = \frac{\partial}{\partial y}\left[\omega(y,t)K_m\psi(\alpha)p(y,t)\right] + \frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2}{\partial y^2}\left[\eta_i D_i\eta_i^T)p(y,t)\right] \quad (19)$$

where $\eta_1 = \omega(y,t)\psi(\alpha)$ and $\eta_2 = K_m\omega(y,t)\psi(0)$. By analogy to other parameters, such as k_1 and d_c, there must exist an analogous FP equation to be solved when applying the independent Gaussian white noise term in SDE. It should be noted that the incident wave angle is assumed to be constant in working examples, which may not be the case on a real coast, especially near a jetty or a barrier, which experience the greatest wave exposure and the most severe sediment transport. A deliberate analysis needs to be performed by incorporating a wave transformation model.

(6) All numerical models have some degree of model imperfection because no empirical equation can exactly reproduce nature. In the present SDE model, storm-induced shoreline change and subsequent beach recovering are assumed to "average out" over the long term, e.g., the breaking wave data are calculated from averaged or statistical offshore wave data. So the emphasis of this paper is on the long-term evolution of alongshore transport rather than on seasonal or storm-based evolution; thus, the forces resulting from wave conditions can only be considered as averages of this force for the ensemble. However, if recorded data are available for a wave storm and are taken as a certain input, predicting the probability density of the shoreline position for one extreme wave storm by considering some other parameters as random coefficients, k_1 or d_c, is not difficult. A theoretical framework for a general stochastic system involving the multi-dimensional sediment transport pathways and multi-variate model parameters is given in Appendix 1.

(7) Additive noise can be incorporated to represent the model error and can be derived from other fluctuations sources without any inconvenience. Actually, the Liouville model can also solve this multiplicative white noise problem (Kubo 1963; Gardiner 2004). A SDE model based on the Liouville equation can be developed for the non-linear random dynamics systems in which the erosion process is based on the one-line type model and the random effects are assumed to be characterised by random variables, which also permit non-Gaussian distributions for the breaking wave parameter. These results will be reported separately.

(8) The solution procedure of the FP equation is the same as the formation with a general second-order parabolic partial differential equation. A second-order or higher-order implicit difference approximation with the backward in time and central in space method can be used.

(9) The simple CERC formula is employed because it provides (general, non-unique) empirical fits to field data sets and is popular in shoreline dynamic models. The ideas presented here may also work with other forms of empirical laws, but these generalisations will not be pursued here.

4 Conclusions

Uncertainty is introduced into the coastal morphological processes for each cell along the coastline by recasting an Itô SDE by considering the stochastic nature of forces and the initial shoreline position. Accordingly, the FP model associated with the Itô SDE is formulated as an explicit expression of the PDF of the shoreline positions. The mean evolutionary dynamics of the time-dependent shoreline position, updated from their PDFs each time increment, is driven by a well-established deterministic framework. The numerical accuracy of the model has been confirmed by comparison of the mean shoreline predicted by the present model with that from the deterministic and Monte Carlo simulations approaches. The effect of the input parameters on the predicted shoreline statistical properties is evaluated through a series of sensitivity analyses. The proposed model is able to account for the effects of uncertainties in the breaking wave parameter and the initial shoreline position and provides an efficient means of predicting the complete stochastic characteristics of shoreline evolution without requiring a Monte Carlo sampling.

Compared to the conventional predictions based solely on deterministic solutions, the analyses here provide a reasonable foundation by admitting uncertainty, and the confidence limits and statistical characteristics of the future shoreline position can be established. The statistics of coastline position are not stationary with respect to either space or time. The present work adopts a simple representation of the breaking wave climate such that the random variability of the breaking wave angle is neglected for the sake of simplicity. A detailed development and application to a few real sites should be encouraged to further account for the spatial variation of the breaking wave height parameters and the multi-dimensional sediment transport pathways.

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Appendix 1: The theoretical framework for a general stochastic system

For a real field site where sediment transport pathways are parallel or shore-normal (both have significant components in two or three dimensions), the simplification made by the one-line alongshore model is unlikely to produce an accurate prediction for the variability of the coastal spatial configurations. In this case, a multi-dimensional deterministic numerical model can be incorporated into this modelling framework, involving a stochastic process of an initial state and some model parameters along these lines. For such a random dynamic coastal system, a more specific model that characterises a Markovian process is often adopted, which is obtained by a decomposition as the similar formulation with Eq. (12):

$$\dot{\mathbf{Y}}(x,t) = \boldsymbol{\kappa}(\mathbf{Y},t) + \mathbf{G}(\mathbf{Y},t)\mathbf{W}(t) \tag{20}$$

where $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_{n_\mathrm{d}})^\mathrm{T}$ is the state vector, n_d is the dimension of the state space, $\boldsymbol{\kappa}(\mathbf{Y},t)$ is a function vector representing the effects of some form of drift, $\mathbf{G}(\mathbf{Y},t)$ is a function vector representing the effects of random diffusion, and $\mathbf{W}(t)$ is a n_e-dimensional Gaussian (or white) noise vector. Assuming that the white noise processes are mutually independent, the co-variance parameter matrix of $\mathbf{W}(t)$ will be a diagonal matrix with its diagonal terms equal to the variance of the white noise processes $W_1(t), W_2(t), \ldots, W_{n_\mathrm{e}}(t)$ and $\mathbf{D} = (D_{11}, D_{22}, \ldots, D_{n_\mathrm{e}n_\mathrm{e}})$.

Equation (20) can also be written in the vector form of the standard Itô equation (see Soong 1973):

$$\mathrm{d}\mathbf{Y}(x,t) = \boldsymbol{\kappa}(\mathbf{Y},t)\mathrm{d}t + \mathbf{G}(\mathbf{Y},t)\mathrm{d}\mathbf{W}(t) \tag{21}$$

with $E\{\mathrm{d}\mathbf{W}(\mathrm{t})\} = \mathbf{0}$ and $E\{[\mathrm{d}\mathbf{W}(\mathrm{t})]^2\} = 2\mathbf{D}\mathrm{d}t$.

The dynamical system described by Eq. (21) with the deterministic operators $\boldsymbol{\kappa}$ and $\mathbf{G}$ driven by some expressions of sediment transport, the evolutionary PDFs of the shoreline position $\mathbf{Y}$, $p(\mathbf{y},\mathrm{t})$, will satisfy the FP equation (Soong 1973; Gardiner 2004):

$$\frac{\partial p(\mathbf{y},t|\mathbf{y}_0,t)}{\partial t} = -\sum_{j=v}^{n_\mathrm{d}} \frac{\partial}{\partial y_v}\left[\eta(y,t)_v p\right] + \frac{1}{2}\sum_{u,v=1}^{n_\mathrm{d}} \frac{\partial^2}{\partial y_v \partial y_u}\left[(GDG^T)_{uv} p\right] \tag{22}$$

The term $(GDG^\mathrm{T})_{uv}$ is given by:

$$(GDG^\mathrm{T})_{uv} = \sum_{k,l=1}^{n_\mathrm{e}} D_{kl} G_{uk}(y,t) G_{vl}(y,t) \quad (u,v = 1,2,\ldots,n_\mathrm{d}), \quad (k,l = 1,2,\ldots,n_\mathrm{e}) \tag{23}$$

where G_{uk} and G_{vl} are components of $G(y,t)$.

The theoretical framework described above is formulated into a stochastic shoreline evolution model to use for a n_d-dimensional SDE with a n_e-dimensional Wiener process. In these working examples, the state function vector is derived using the one-line model, which means that only longshore transport is considered to contribute to the shoreline changes, implying $n_\mathrm{d} = 1$. For simplicity's sake, only the characteristic wave height parameter is taken as a random parameter, implying $n_\mathrm{e} = 1$.

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